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THE PARAMETRIZED STRONGLY IMPLICIT METHOD FOR SOLVING ELLIPTIC DIFFERENCE EQUATIONS

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SUMMARY

A new method for solving elliptic difference equations is derived based on the strongly implicit method. This parametrized strongly implicit method has three free parameters which may be functions of the field's nodal point. The method has some resemblance to the SOR techniques, but in the present method the off-diagonal entries are also over-relaxed. The main application of this method is for transport equations such as those governing the fluid flow and heat transfer fields.

KEY WORDS Matrix Iterative Methods Strongly Implicit Procedure.

INTRODUCTION

A numerical approach to solving an elliptic partial differential equation using finite difference techniques leads to the sparse algebraic system

$$[A]\{\Phi\} = \{D\} \tag{1}$$

where $\{\Phi\}$ is an m-element variable vector, $[A]$ is an $m \times m$ coefficients matrix and $\{D\}$ is an m-element source vector, where m is the number of grid points spread over the spetial domain of solution (say $m = m_1 \times m_2$ for a rectangular domain). Usually, for transport phenomena, $[A]$ is a non-symmetric matrix. A direct solution of equation (1) is very inefficient sometimes even for symmetric matrices $[A]$, and most of the time cannot be applied since usually m is very large. For large systems, equation (1) is usually solved by iterative procedures. The iterative procedures are very commonly divided into two classes: (1) the basic iterative techniques such as SOR, ADI,[1] strongly implicit (SI)[2] and modified strongly implicit (MSI)[3] among others; (2) the acceleration techniques where gradient methods are used to solve equation (1) iteratively; these methods are usually preconditioned by some incomplete LU decomposition or by any of the procedures in class (1).[5–12] Usually the methods of class (2) are faster than those from class (1). However, for a non-symmetric matrix $[A]$ which is also blocked (meaning that we have to solve more than one variable at any grid point of the field) the rate of convergence of the acceleration methods slows down dramatically.[4] Therefore the basic iterative methods are still widely used, and thus it is important, if it is possible, to improve them.

In the past we have presented a modification to the classical SI method,[3] in which, in addition to the standard algorithm, the variables are solved implicitly along the diagonals of the computational domain. This algorithm is a general modification of the SI technique,[13] which improves the rate of convergence. One of the reasons for the slow rate of convergence of iterative techniques of the SI procedure type is due to the asymmetry of $[A]$. The idea in the present study is that the rate of

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convergence might be increased by inserting certain free parameters into the SI and the MSI schemes' coefficients, in such a way that the weighting between the implicit and the lagging parts in the algorithm can be optimized, leading to a higher rate of convergence. This idea is very similar to the SOR method, which improves the classical one point iteration methods (such as the Gauss–Siedel (GS) and the Jacobi iteration techniques among others). This new method, the parametrized strongly implicit (PSI) algorithm, will be described and discussed in the present paper.

THE PARAMETRIZED SI (PSI) ALGORITHM

Considering the solution of equation (1), let $(i,j)[1 \leqslant i \leqslant m_1, 1 \leqslant j \leqslant m_2]$ be the indices of any discrete points in the two dimensional computational domain, then the variable $\Phi_{i,j}$ at any point (i,j) can be calculated from the following recursive algorithm:

$$\Phi_{i,j}^{(n)} = b_{i,j}\Phi_{i,j+1}^{(n)} + c_{i,j}\Phi_{i+1,j}^{(n)} + d_{i,j} + \beta_{ij}\Phi_{i,j-1}^{(n-1)} + \gamma_{i,j}\Phi_{i+1,j}^{(n-1)} + \alpha_{i,j}\Phi_{i,j}^{(n-1)} \tag{2}$$

where n is the iteration level index, and $T=([B],[C],[d],[\alpha],[\beta],[\gamma])$ is a group of two dimensional coefficient matrices. Equation (2) presents a general family of procedures to solve equation (1) iteratively, where the elements of the coefficients matrices in T are related to the entries of the matrix $[A]$. The equation presents, in general, the PSI algorithm, where the standard SI procedure can be derived by assuming

$$[\alpha] = [\beta] = [\gamma] = 0 \tag{3}$$

In order to relate the iterative scheme, equation (2), to the equation to be solved, we have to write the nodal relation for equation (1):

$$E_{i,j}\Phi_{i+1,j} + N_{i,j}\Phi_{i,j+1} + W_{i,j}\Phi_{i-1,j} + S_{i,j}\Phi_{i,j-1} + P_{i,j}\Phi_{i,j} + V_{i,j} = 0 \tag{4}$$

where $[E],[W],[N],[S]$ and $[P]$ are known coefficients matrices, and $[V]$ is the source term. The relation between the algorithm coefficients in equation (2) and the algebraic system to be solved, equation (4), can be obtained by assuming that all the elements participating in this equation are under or over-relaxed:

$$\begin{aligned}&(1-\varepsilon_E)E_{i,j}\Phi_{i+1,j}^{(n)} + (1-\varepsilon_N)N_{i,j}\Phi_{i,j+1}^{(n)} + (1-\varepsilon_W)W_{i,j}\Phi_{i-1,j}^{(n)} + (1-\varepsilon_S)S_{i,j}\Phi_{i,j-1}^{(n)}\\ &\quad + (1-(1-\varepsilon_P)P_{i,j}\Phi_{i,j}^{(n)} + \varepsilon_E E_{i,j}\Phi_{i+1,j}^{(n-1)} + \varepsilon_N N_{i,j}\Phi_{i,j+1}^{(n-1)} + \varepsilon_W W_{i,j}\Phi_{i-1,j}^{(-1)}\\ &\quad + \varepsilon_S S_{i,j}\Phi_{i,j-1}^{(n-1)} + \varepsilon_P P_{i,j}\Phi_{i,j}^{(n-1)} + V_{i,j} = 0\end{aligned} \tag{5}$$

where $\varepsilon_E, \varepsilon_W, \varepsilon_N, \varepsilon_S$ and ε_p are predetermined parameters which may be function of the point (i,j). As these parameters are different from zero, only a fraction of every term appearing in equation (4) is treated in the new iteration level n. Substitution of the PSI algorithm, equation (2), into equation (5) will give the following relations between the various coefficients' matrices:

$$F_{i,j} = \frac{1}{(1-\varepsilon_P)P_{i,j} + \left(\dfrac{W_{i,j}}{1-\alpha_{i-1,j}}\right)c_{i-1,j} + \left(\dfrac{S_{i,j}}{1-\alpha_{i,j-1}}\right)b_{i,j-1}} \tag{6a}$$

$$b_{i,j} = F_{i,j}N_{i,j}(1-\varepsilon_N) \tag{6b}$$

$$c_{i,j} = F_{i,j}E_{i,j}(1-\varepsilon_E) \tag{6c}$$

$$\alpha_{i,j} = F_{i,j}\left[P_{i,j}\varepsilon_P + \frac{W_{i,j}}{1-\alpha_{i-1,j}}\gamma_{i-1,j} + \frac{S_{i,j}}{1-\alpha_{i,j-1}}\beta_{i,j-1}\right] \tag{6d}$$

$$\beta_{i,j} = F_{i,j}\varepsilon_N N_{i,j} \tag{6e}$$

$$\gamma_{i,j} = F_{i,j}\varepsilon_E E_{i,j} \tag{6f}$$

$$d_{i,j} = F_{i,j}\left\{V_{i,j} + \frac{W_{i,j}}{1-\alpha_{i-1,j}}[(b_{i-1,j}+\beta_{i-1,j})\Phi^{n-1}_{i-1,j+1} + d_{i-1,j}]\right.$$
$$\left. + \frac{S_{i,j}}{1-\alpha_{i,j-1}}[(c_{i,j-1}+\gamma_{i,j-1})\Phi^{n-1}_{i+1,j-1} + d_{i,j-1}]\right\} \tag{6g}$$

where

$$\varepsilon_S = \frac{\alpha_{i,j-1}}{\alpha_{i,j-1}-1} \tag{6h}$$

$$\varepsilon_W = \frac{\alpha_{i-1,j}}{\alpha_{i-1,j}-1} \tag{6i}$$

Thus, the requirement that the present algorithm, equation (2), will resemble the SI procedure reduces the number of free parameters from five to three. The standard SI procedure can be recovered by assuming

$$\varepsilon_N = \varepsilon_E = \varepsilon_P = 0 \tag{7}$$

The nature of the PSI algorithm may be understood in the following way.

The system matrix $[A]$ has the following form

$$[A] = [L] + [M] + [U] \tag{8}$$

where $[L]$ and $[U]$ are strictly lower and upper matrices defined by

$$L_{i_1,i_2} = \begin{cases} W_{i,j}, \text{ if } i_i = i+(m_1-1)j, i_2 = i-1+(m_1-1)j \\ S_{i,j}, \text{ if } i_1 = i+(m_1-1)j, i_2 = i-1+(m_1-2)j \\ 0, \quad \text{elsewhere} \end{cases} \tag{9a}$$

$$U_{i_1,i_2} = \begin{cases} E_{i,j}, \text{ if } i_1 = i+(m_1-1)j, i_2 = i+1+(m_1-1)j \\ N_{i,j}, \text{ if } i_1 = i+(m_1-1)j, i_2 = i+1+m_1 j \\ 0, \quad \text{elsewhere} \end{cases} \tag{9b}$$

$$M_{i_1,i_2} = \begin{cases} P_{i,j}, \text{if } i_1 = i_2 = i+(m_1-1)j \\ 0, \text{elsewhere} \end{cases} \tag{9c}$$

Constructing the PSI algorithm, we first define three two-dimensional matrices $[\alpha]$, $[\beta]$ and $[\gamma]$ with the restriction

$$[\alpha_{i,j} \text{ and } \gamma_{i,j}](i = m_1, 1 \leqslant j \leqslant m_2) = [\beta_{i,j} \text{ and } \alpha_{i,j}](1 \leqslant i \leqslant m_1, j = 1) = 0 \tag{10}$$

In order to study the scheme formulated in equations (2) and (6), let us define the matrix $[H]$ to be

$$(H\Phi)_{i,j} = -\frac{1}{F_{i,j}}(\alpha_{i-1,j}\Phi_{i-1,j} + \alpha_{i,j-1}\Phi_{i,j-1} + \beta_{i,j}\Phi_{i,j+1} + \gamma_{i,j}\Phi_{i+1,j}) - P_{i,j}\varepsilon_P\Phi_{i,j} \tag{11}$$

A resemble of the under/over-relaxation feature of the PSI algorithm can be achieved by adding equation (11) to equation (1)[or equation (4)], defining the following iteration procedure:

$$[N]\Phi^n = [H]\Phi^{n-1} + D \tag{12a}$$

where

$$[N] = [A] + [H] \tag{12b}$$

Following equation (9), the matrix $[N]$ has the following structure:

$$[N] = [L_N] + [\Delta] + [U_N] \tag{13}$$

where

$$L_{N_{i1,i2}} = \begin{cases} W_{i,j} - \dfrac{\alpha_{i-1,j}}{F_{i,j}}, & \text{if } i_1 = i-(m_1-1)j, i_2 = i-1-(m_1-1)j \\ S_{i,j} - \dfrac{\alpha_{i,j-1}}{F_{i,j}}, & \text{if } i_1 = i-(m_1-1)j, i_2 = i-1+(m_1-2)j \\ 0, & \text{elsewhere} \end{cases} \tag{14a}$$

$$U_{N_{i1,i2}} = \begin{cases} E_{i,j} - \dfrac{\gamma_{i,j}}{F_{i,j}}, & \text{if } i_1 = i-(m_1-1)j, i_2 = i+1-(m_1-1)j \\ N_{i,j} - \dfrac{\beta_{i,j}}{F_{i,j}} & \text{if } i_1 = i-(m_1-1)j, i_2 = i-1+(m_1-2)j \\ 0, & \text{elsewhere} \end{cases} \tag{14b}$$

and

$$\Delta_{i_1 i_2} = \begin{cases} A_{i_1,i_2}(1-\varepsilon_P), & \text{if } i_1 = i_2 \\ 0, & \text{elsewhere} \end{cases} \tag{14c}$$

where ε_P may be a function of the nodal point i, j.

The next step is similar to what has been done in References 1 and 3, where a $[G]\Phi$ term was added to the iteration (12a), in order to make the left hand side of equation (12a) factorable in the following way:

$$([N] + [G])\Phi^n = ([H] + [G])\Phi^{n-1} + D \tag{15}$$

where $[G]$ has the following feature:

$$[N] + [G] = ([L_N] + [\Delta])[\Delta]^{-1}([\Delta] + [U_N]) \tag{16}$$

and $[\Delta]$ is a diagonal matrix.

Considering the right hand side of equation (16), we assume that the $[H] + [G]$ matrix can be defined as the following sum:

$$[H] + [G] = [T] + [Q] \tag{17}$$

where $[T]$ is defined as

$$[T] = ([L_N] + [\Delta])[\Delta]^{-1}([\Delta_T] + [U_T]) \tag{18}$$

where $[\Delta_T]$ is a diagonal matrix and $[U_T]$ is strictly upper diagonal matrix. As it turns out, the PSI procedure has enough free parameters, so that equation (18) can be satisfied together with the following definitions of the $[U_T]$, $[Q]$ and $[\Delta_T]$ matrices:

$$\Delta_{T_{k,k[k=i+(j-1)m_1]}} = \alpha_{i,j} \tag{19a}$$

$$U_{T_{i_1,i_2}} = \begin{cases} \beta_{i,j}, & \text{if } i_1 = i-(m_1-1)j, i_2 = i+1-(m_1-1)j \\ \gamma_{i,j}, & \text{if } i_1 = i-(m_1-1)j, i_2 = i+1+m_1 j \\ 0, & \text{elsewhere} \end{cases} \tag{19b}$$

$$Q_{i_1,i_2}=\begin{cases}(W_{i,j}+\alpha_{i-1,j})(B_{i-1,j}+\beta_{i-1,j}), & \text{if } i_1=i-(m_1-1)j, i_2=i+1-(m_1-2)j\\ (S_{i,j}+\alpha_{i,j-1})(C_{i,j-1}+\gamma_{i,j-1})_{i,j}, & \text{if } i_1=i-(m_1-1)j, i_2=i+1+m_1 j\\ 0, & \textit{elsewhere}\end{cases} \tag{19c}$$

From equations (14)–(19) we can derive the following two steps per iteration that the PSI algorithm defines:

Step 1:

$$([L]+[\Delta])[\Delta]^{-1}g=D+[Q]\Phi^{n-1} \tag{20a}$$

Step 2:

$$([\Delta]+[U_N])\Phi^n=g+([\Delta_T]+[U_T])\Phi^{n-1} \tag{20b}$$

where g is an intermediate vector of results. Again it can be seen that for $[\Delta_T]=[U_T]=0$ the standard SI procedure is recovered.

The PSI algorithm will converge if the error series $e^n=\Phi^{(n)}-\Phi^{(\text{exact})}$ converges to zero. The iteration matrix $[B]$ defined by

$$e^{(n)}=[B]e^{(n-1)} \tag{21}$$

can be computed for the PSI algorithm to be

$$[B]=([\Delta]+[U_N])^{-1}\{[\Delta]([L]+[\Delta])^{-1}[Q]+[\Delta_T]+[U_T]\} \tag{22}$$

and the only condition for that algorithm to be convergent is[12] that λ the spectral radius of $[B]$, should be less than 1. Also the rate of convergence of the algorithm, ρ, is a monotonic function of λ, defined by

$$\rho=-\ln(\lambda) \tag{23}$$

Therefore, it is obvious that as λ becomes smaller, the rate of convergence is higher. Thus, theoretically we can maximize ρ with respect to the three free parameters of the scheme. However, trying to check the spectral radius for the standard SI algorithm,[8] even for very simple problems (such as $\nabla^2\Phi=0$ on an equal spaced grid) is extremely difficult. Therefore it can be understood why it is almost impossible to carry out the optimization for the PSI algorithm. Hence, in the present paper, we present some special cases, for which we try to optimize the rate of convergence of the PSI procedure. Although it is not general, we can get some feeling about the near optimal procedure's parameters, which might lead to more general conclusions.

EXAMPLES

In this section, some simple examples and special cases will demonstrate the use and the advantages of the PSI technique. We do not expect that these examples will lead us to a definite conclusion about the method, but they might have some effect on the choice of the algorithm's free parameters.

Case 1: $\varepsilon_E=\varepsilon_N=0$

This case is very similar to the successive over-relaxation technique. Here $[H]$ is a lower matrix and $[U_T]=0$ and $[U_N]=[\mathbf{U}]$. Therefore the iteration matrix is

$$[B]=([\Delta]+[U])^{-1}\{[\Delta]([L]+[\Delta])^{-1}[Q]+[\Delta_T]\} \tag{24}$$

If ε_P is not a function of the nodal point, then the spectral radius is given by the following equation:

$$\lambda([L]+[\Delta])[\Delta]^{-1}\left\{[\Delta]\left[(1-\varepsilon_P)+\frac{\varepsilon_P}{\lambda}\right]+[U]\right\}Y=[G]Y \tag{25}$$

where Y is the eigenvector belonging to λ. For the case where $\varepsilon_P=0$ we get the standard SI procedure with the spectral radius of

$$\mu=\lambda(\varepsilon_P=0) \tag{26}$$

Equation (25) has been solved numerically for example 1 of Reference 5. Figure 1 describes the rate of convergence ratio

$$K=\frac{\ln\lambda}{\ln\mu}$$

as a function of ε_P. For this Laplace equation example, Figure 2 shows the residuals in the L_2 norm as function of the number of iterations for two different grids. It can be seen that the ratio K is the same for both grids. It is interesting to note that after many iterations (when the residuals are small), the PSI algorithm produces a periodic-like behaviour for the residuals, whereas the residuals of the SI algorithm still converge monotonically. The main conclusion from this case study is that for every $\varepsilon_P>0$, the PSI procedure is better than the corresponding SI procedure.

Case 2: $\varepsilon_P=0$, ε_E *and* $\varepsilon_N\neq 0$

In this case the only coefficients that are going through the under/over-relaxation process are those of the non-central point. For the simple case where

$$\varepsilon_E=\varepsilon_N=\varepsilon$$

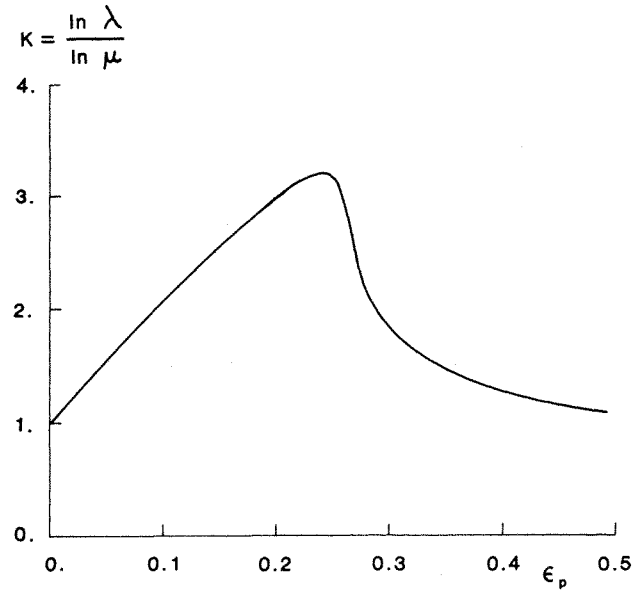

Figure 1. Rate of convergence for case 1

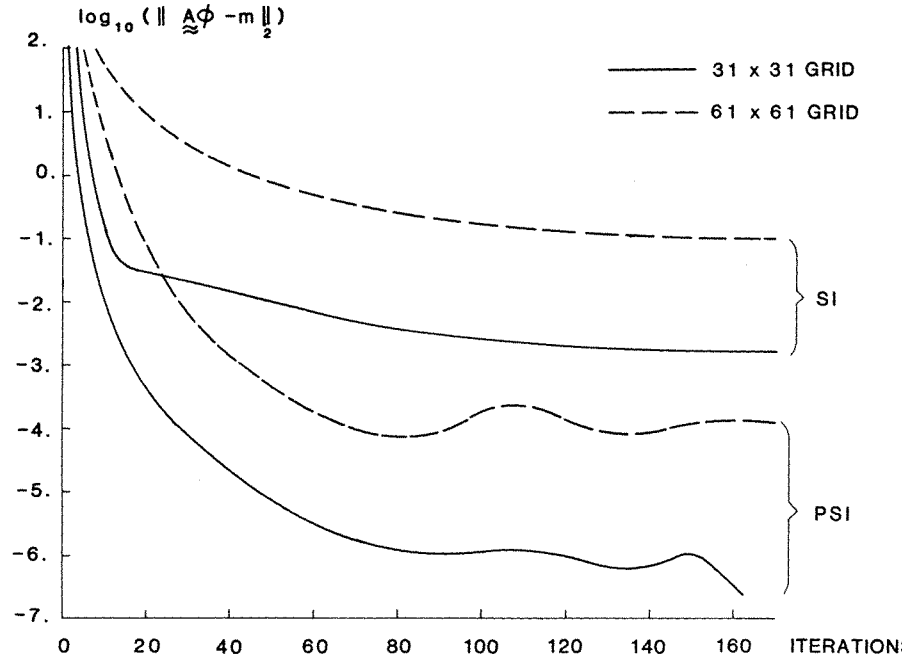

Figure 2. Variation of the L_2 error norm of SI and PSI techniques with iterations

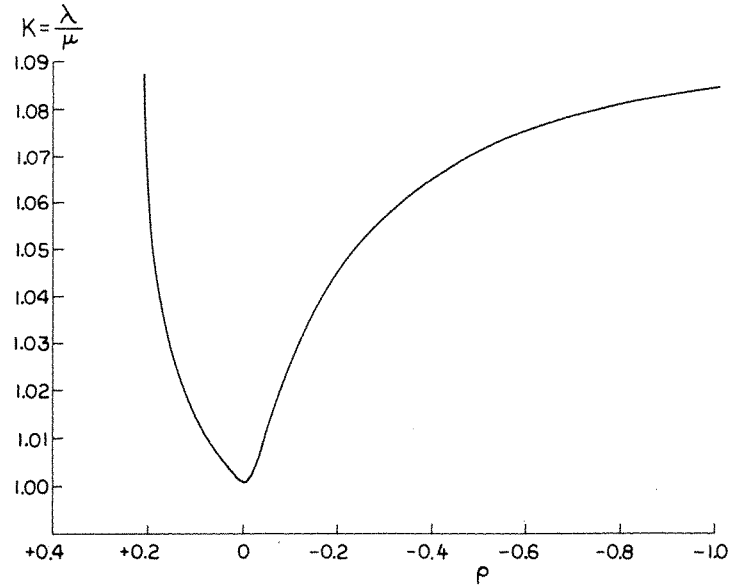

Figure 3. The spectral radius for case 2 as a function of $\varepsilon_E = \varepsilon_N = \rho$

the equation for the spectral radius is

$$\lambda([L]+[\Delta])[\Delta]^{-1}\left\{[\Delta]+[U]\left(1-\varepsilon+\frac{\varepsilon}{\lambda}\right)\right\}Y=[G]Y \tag{27}$$

where Y is, again, the eigenvector. This equation was solved for the same example as in case 1, with 41×41 points. Figure 3, which presents the ratio λ/μ shows that, as ε differs from 0, the SI procedure is better than the PSI procedure for $\rho \neq 0$. This should not be a surprise since also for other methods[1] (such as, for instance, the ADI method) it is known that their performance cannot be improved when only the off-diagonal entries of $[A]$ are under/over-relaxed. However, this does not mean that one cannot find such an ε that together with $\varepsilon_P \neq 0$ will give a rate of convergence which is higher than when $\varepsilon = 0$.

SOME REMARKS ON OTHER POSSIBLE PSI PROCEDURES

The study in the last section tends to indicate that the only possibility to get a rate of convergence with the PSI procedure which is faster than that of the SI procedure is by under/over-relaxation at least the diagonal terms of $[A]$. In addition to the algorithm presented in the first section, it is possible to formulate some other PSI-like procedures, two of which follow. It can be seen, from equation (20), that the SI algorithm can be written as

$$([L]+[\Delta])[\Delta]^{-1}([\Delta]+[U])\Phi^n = D+[Q]\Phi^{n-1} \tag{28}$$

Such algorithms can be formulated by taking ω times one of the following equations

$$[I]\Phi^n = [I]\Phi^{n-1} \tag{29a}$$

$$([L]+[\Delta])[\Delta]^{-1}\Phi^n = ([L]+[\Delta])[\Delta]^{-1}\Phi^{n-1} \tag{29b}$$

and adding it to $1-\omega$ times equation (28). ω can be a function of the nodal point (i,j). The iteration matrices for those two cases are:

$$[B_1] = ([\Delta]+[U]+\omega([L][\Delta]^{-1}+[I]-[\Delta]-[U]))^{-1}[\Delta]([L]+[\Delta])^{-1}\{[G]+\omega([I]-[G])\} \tag{30a}$$

$$[B_2] = \{[\Delta]+[U]-\omega([\Delta]+[U]-[I])\}^{-1}[\Delta]([L]+[\Delta])^{-1}\{[G]+\omega([L][\Delta]^{-1}+[I]-[G])\} \tag{30b}$$

for equations (29a) and (29b) respectively. It should be noted that the $[B_2]$ iteration matrix involves also parametrization of the off-diagonal entries of $[A]$. Since the optimal ω cannot be found analytically, these schemes were tested on the Laplace equation. It has been found that for the $[B_1]$ case, the optimal ω was around $\omega_1 \approx 0{\cdot}38$ with $\ln\lambda_1/\ln\mu \approx 2{\cdot}6$, and for the $[B_2]$ case, the optimal ω was obtained for $\omega_2 \approx 0{\cdot}46$ with $\ln\lambda_2/\ln\mu \approx 2{\cdot}4$. Thus, it might be possible to get a higher rate of convergence in cases when all the three parameters are under/over-relaxed than in case when only the diagonal parameter $\varepsilon_P \neq 0$.

To conclude this study, a test case with non-symmetric $[A]$ was chosen: the incompressible two dimensional steady state flow inside a driven cavity as is describe in Figure 4, which is a well known problem for testing numerical schemes in fluid dynamics.[14]

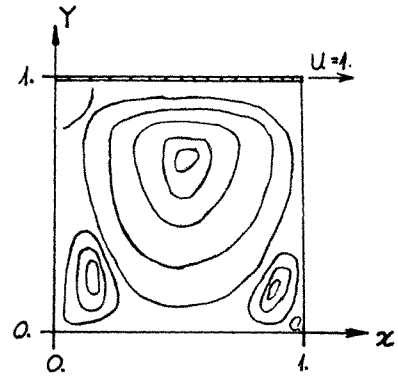

Figure 4. Example of the driven cavity field

The governing equations are the following (ψ ξ) relations:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\xi=0 \tag{31a}$$

$$\frac{\partial(u\xi)}{\partial x}+\frac{\partial(v\xi)}{\partial y}=\frac{1}{R}\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}\right) \tag{31b}$$

where the velocities $u(x, y)$, $v(x, y)$ are defined by

$$u=\frac{\partial\psi}{\partial y} \tag{31c}$$

$$v=-\frac{\partial\psi}{\partial x} \tag{31d}$$

and where ψ is the flow stream function, ξ is the flow vorticity and R is the flow Reynolds number. The solutions for this problem are well established and discussed in many papers.[4] Since equation (31a) is of the Poisson type, then the optimum situation is similar to that of case 1 : $\varepsilon_E = \varepsilon_N = 0$. Since two variables are involved, ε_P will now be a vector defined by $\varepsilon_P = (\omega_1, \omega_3)^T$ for the vector (ψ, ξ) in this equation. On the other hand, equation (31b) is like a one variable equation and we will assume that all the three parameters are different from zero since it is a convection–diffusion-like equation. We will assume furthermore that $\varepsilon_P = \omega_4$ and $\varepsilon_N = \varepsilon_E = \omega_2$ for this equation. Also we will assume that all the parameters do not depend on the nodal point (i, j) but they are sensitive to the Reynolds number. This means that as $R \to 0$, $\omega_3 \to \omega_1$, and $\omega_4 \to 0$.

Figure 5 presents the variation of some optimal parameters and the optimal rate of convergence as functions of the Reynolds number, with the assumption that $\omega_1 = \omega_3$ and $\omega_4 = 0$. The last assumption means that the diagonal of equation (31a) is over-relaxed with ω_1 and the off-diagonal terms of equation (31b) are over-relaxed with ω_2. Figure 5 depicts the variation of the ratio of the optimal parameters ω_2/ω_1 as functions of the Reynolds number for three cases. The first is the PSI algorithm with the above assumptions; the second two cases are the two methods described by equations (30) where ω_1 is used for equation (31a) and ω_2 is used to over-relaxed equation (31b). It can be seen that all these cases are superior to the SI procedure, and for a transport equation which is convection dominant, the optimum rate of convergence might be achieved for ε_E and $\varepsilon_N \neq 0$. It

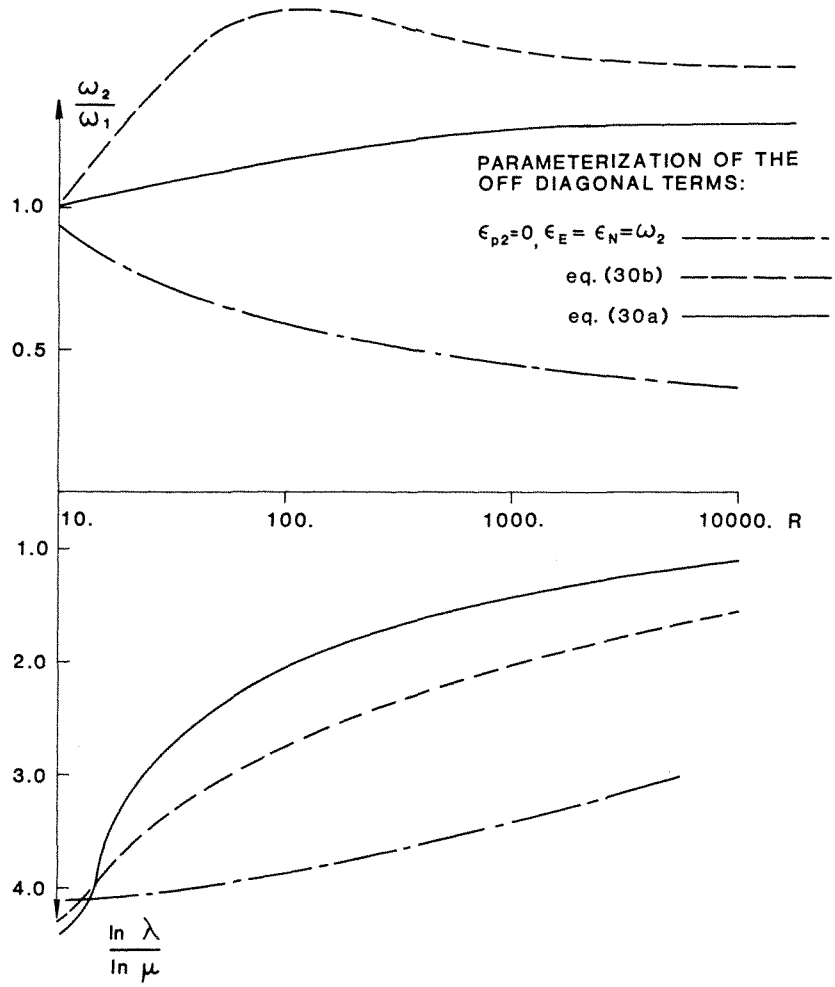

Figure 5. Rate of convergence and the optimal parameter for various Reynolds numbers of the driven cavity

should be noted that for all these methods, as $R \to \infty$, the rate of convergence tends to that of the SI procedure.

CONCLUSIONS

In the light of Stone's[2] suggestion that the rate of convergence of the SI procedure might be optimized by inserting some free parameters into the algorithm, the present study suggests a family of algorithms of the SI procedure type with a maximum of three free parameters. It has been shown that for pure diffusion problems the highest rate of convergence occurs when parametrizing only the diagonal coefficients of the system. Here, any positive entry for this parameter will cause a faster convergence than that of the SI procedure. A maximum rate of convergence for non-symmetric cases is obtained when the off-diagonal coefficients are also parametrized. The reason for the success of the PSI procedure is the same as that for the various SOR techniques. The main difference between applying the SOR idea to techniques which are implicit along lines (such as the ADI method and the line relaxation method) and applying it to the SI technique is that all the line-

implicit techniques are like one dimensional iterative procedures and the application of under/over-relaxation to them is much more easy then for the SI technique. That is because the SI technique is a two dimensional iterative procedure and application of the SOR idea directly to the algorithm will not be a benefit. We have found that the right way to do it is through the PSI algorithm where also the off-diagonal terms are under/over-relaxed. Applying the PSI procedure without parametrizing the off-diagonal coefficients to systems which are convection dominant, will give almost the same rate of convergence as that of the SI. On the other hand some computational experiments have shown that high rate of convergence can be obtained for high Reynolds number flows when attention is given to over-relaxing also the off-diagonal system's coefficients. The present paper comes basically to present the PSI algorithm and to describe some preleminary study of it. Obviously, some more study has to be done before making any final conclusions in comparing it to other iterative methods of the same type.

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